Perspective

Perspectives in Multiparametric Programming and Explicit Model Predictive Control

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Introduction

In recent years, the availability and use of model-based optimization technologies for improved design, operation and control of various types of engineering systems has significantly increased and is typically provided by commercial tools, such as gPROMS® (PSE, Ltd., 2008), Matlab® (The MathWorks, 2008), GAMS®, and ACM® (AspenTech, 2008). Despite these advances, however, one major difficulty for the widespread application of optimization to real systems and processes arises from the unavoidable presence of variability in the model/process parameters, such as fluctuations in inputs and measurements, variations in inherent system properties and characteristics, uncertainty in prices and availability of resources and the like. In an optimization framework, such variations typically result in deviations from an ideal/prescribed optimal point, thus, either failing to fully exploit the benefits of the optimization solution or requiring the repetitive solution of the problem for different values of the changing conditions.

Multiparametric programming is an optimization technology that allows determining in a computationally efficient way the optimal solution (profile) of an optimization problem as a function of the varying parameters—without exhaustively enumerating the entire space of parameters. In this way, the repetition of the optimization problem solution can be avoided, while the optimal solution can readily be updated, from the multiparametric pre-computed solution map. In the context of model predictive control (MPC) (or real-time/online optimization), multiparametric programming can be effectively used to obtain the optimal control inputs as an explicit function of the state variables/measurements. This is the notion of explicit or multiparametric model predictive control—the "MPC-on-a-chip" technology (see cover figure). These concepts are briefly discussed next.

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Multiparametric programming and model predictive control

Traditional model predictive control (MPC) aims to provide a sequence of control actions/inputs over a future time horizon, which seeks to optimize the controller performance based on the predicted states of the system. This is achieved by repetitively solving an online optimization problem, which describes the (past, present, future) behavior of the system. The basic idea of the MPC implementation is shown in Figure 1, where at the current time interval the optimization problem is solved to minimize the state and control deviations from the set point, by implementing the optimal values of the control/input variables. Note that only the first control element is implemented and this sequence is repeated at the next time interval, for the new state measurements or estimates, until the desired or set point values are obtained. The key advantage of MPC is that it is model-based, and it can take into account the constraints on the state and control variables. A key limitation is its online computational effort, due to the repetitive solution of the underlying optimization problem. It is also worth noticing the "implicit" nature of traditional MPC—only the optimal values of the control action are numerically determined at the specific values of the observed/measured states without any knowledge, physical or mathematical, of the governing control law(s).

In a multiparametric programming setting, a typical solution of an optimization problem, involving parameters that vary within lower and upper bounds, is shown in Figure 2. Note that the optimal solution comprises a finite set of regions (called critical regions), where a particular solution is valid, along with analytical expressions that relate the optimization (decision) variables as a function of the varying parameters. The procedure for obtaining the control solutions and corresponding critical regions depends on whether the functions in the optimization problem are linear, quadratic, nonlinear, convex, differentiable or not, as well as whether the decision variables and parameters are continuous, binary, time-varying or not—i.e., the type of mathematical model involved. Table 1

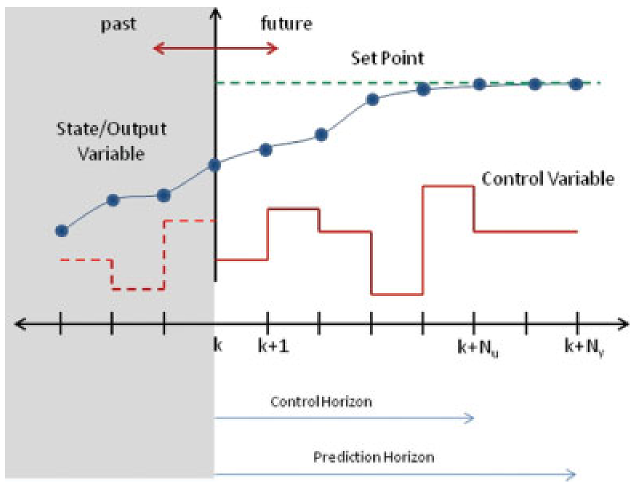


Figure 1. Model predictive control implementation.

provides a classification of the various classes of multiparametric programming problems, together with some milestone developments.

The implications for model predictive control (MPC) are then rather clear. Multiparametric programming can be used to obtain the control inputs (the optimization variables) as an explicit function of the states (the varying parameters)—these are in fact the governing control laws of the system at hand! This reduces the online model-based control and optimization problem to a sequence of function evaluations—this is termed multiparametric or explicit model predictive control (mp-MPC). In contrast to the classical MPC and reflecting its explicit nature, mp-MPC provides valuable insights to the fundamental understanding of the (inherent) control performance of the system. These parametric control profiles (laws) can then be stored in simple computational hardware, such as a microchip. The concept of replacing the online optimization via the exact mapping of its optimal multiparametric solution is termed "online optimization via offline optimization" (Pistikopoulos, 1997, 2000), while the ability of mp-MPC to be implemented on the simplest possible hardware is denoted as "MPC-on-a-chip" technology (Pistikopoulos et al., 2004, 2008)—as illustrated in the cover figure. Table 2 provides a classification of the various classes of explicit MPC problems, together with a number of milestone developments. The advances in multiparametric programming and control are the subject of a two-volume book edition by Pistikopoulos et al. (2007a,b).

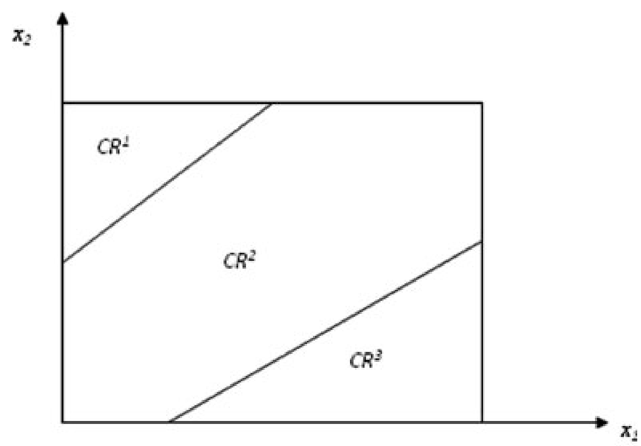
The key advantages of the MPC-on-a-chip implementation are that (1) it is computationally efficient since it requires simple function evaluations, (2) it does not require any online optimization software, (3) its explicit form makes mp-MPC ideal for safety critical applications, and (4) allows for advanced model-based controllers to be implemented in portable and/or embedded devices. This has paved the way for many advanced control applications in chemical, energy, automotive, aeronautical, and biomedical systems, among others.

In the next sections, we briefly present a summary of recent/ongoing developments in multiparametric programming, multiparametric/explicit MPC and their applications, as well as highlight key future directions for this exciting research field. We also briefly describe a framework for the systematic development of multiparametric programming and control solutions. From a chemical engineering perspective, we highlight that distinct possibilities for the wide applicability of the proposed MPC-on-a-chip technology are emerging, by taking full advantage of the enhanced understanding of the fundamental phenomena and underlying principles (through a corresponding validated high-fidelity mathematical model)—without making any compromises when it comes to the design of an appropriate advanced controller. From a general control theory viewpoint, we argue that multiparametric programming

(a) Problem formulation

$$z(x) = \min_{u} \; f(u, x)$$
$$\text{s.t.} \quad h(u, x) = 0$$
$$g(u, x) \leq 0$$
$$u \in \Re^{n}$$
$$x \in \Re^{s}$$

(b) Critical Regions



(c) Multi-parametric Solutions

$$u(x) = \begin{cases} u^1(x) & \text{if} \quad x \in CR^1 \\ u^2(x) & \text{if} \quad x \in CR^2 \\ u^3(x) & \text{if} \quad x \in CR^3 \end{cases}$$

Figure 2. Multiparametric programming: graphical interpretation—(a) Problem formulation, (b) critical regions, and (c) multiparametric solution.

Table 1. Milestones in Multiparametric Programming

Multi-Parametric Linear Programming (mp-LP)	Gal and Nedoma (1972); Acevedo (1996); Dua (2000)
Multi-Parametric Quadratic Programming (mp-QP)	Townsley and Candler (1972); Dua (2000)
Multi-Parametric Nonlinear Programming (mp-NLP)	Fiacco (1976); Acevedo (1996)
Multi-Parametric Dynamic Optimization (mp-DO)	Sakizlis (2003)
Multi–Parametric Global Optimization (mp–GO)	Fiacco (1990); Dua et al. (2004)
Multi–Parametric Mixed Integer Linear Programming (mp–MILP)	Marsten and Morin (1975) Acevedo (1996); Kosmidis (1999); Dua (2000)
Multi–Parametric Mixed Integer Quadratic and non-Linear Programming (mp–MINLP)	McBride and Yorkmark (1980); Dua (2000)

and explicit MPC is paving the way toward new application areas and novel implementations of MPC which were not previously possible—the "beyond process control" concept (Morari, 2002).

Multiparametric Programming Theory—Recent Developments and Future Directions

While major advances in the theory of multiparametric programming have been made, especially for linear and quadratic problems (see Pistikopoulos et al., 2007a), much remains to be done, especially for nonlinear and dynamic systems. In this section, we briefly outline a number of recent developments in this direction.

Nonlinear and continuous-time multiparametric programming (mp-NLP, mp-DO)

If the optimization model is static (time-invariant) and involves continuous variables and nonlinear functions, the resulting multiparametric problem is denoted as a multiparametric nonlinear program (mp-NLP). Initial efforts for the solution of mp-NLP focused on the construction of outer/linear approximations, with which a sequence of linear programming (mp-LP) approximations leads toward the solution of the original nonlinear problem, within a prescribed approximation error (Dua V, 2000). Recently, a geometric, vertex-based algorithm was proposed (Narciso, 2009), where the key idea is to partition the parameter space, while at the same time obtaining piecewise affine approximations of the mp-NLP solution—quadratic instead of linear-based approximations can also be used.

If the underlying model is time-varying, then this gives rise to a multiparametric dynamic optimization (mp-DO). For the special class of dynamic models involving linear ordinary differential equations (ODEs), Sakizlis (2003) presented an approach to derive the explicit nonlinear time-varying continuous-time control laws. The significance of this work is that the number of critical regions for the time-varying case is significantly lower than the corresponding discrete-time approximation resulting by considering a fixed number of time intervals—albeit the multiparametric solution becomes nonlinear. Nevertheless, this important result establishes the need for further theoretical developments in multiparametric dynamic optimization (mp-DO) problems.

Global optimization in multiparametric programming

In general mathematical terms, multiparametric programming problem can be viewed as a special class of semi-infinite programming (Mitsos et al., 2008) which even for linear models may become nonlinear and nonconvex, thereby requiring a global optimization approach for their solution. Dua et al., (2004) established that an important feature for the global optimization solution of multiparametric programs is the need not only to establish valid lower bounds (i.e., underestimators), but also valid upper bounds (overestimators), similar to the case of semi-infinite programming. They presented a number of ways of how this can be accomplished for certain classes of nonconvex, nonlinear multiparametric models, such as bilinear.

Bilevel/multilevel and hierarchical programming problems are important classes of optimization problems, with applications in hierarchical decision making, game theory, control, transportation systems and financial systems, among others. They correspond to hierarchical optimization problems, where an optimization problem at a higher level is constrained by another (or more) optimization problem(s) at lower level(s). Even for the simplest case of bilevel linear programs, the solution of such problem requires the use of global optimization. Faísca (2008) has shown that this class of optimization problems can be addressed very effectively by employing multiparametric programming, as a global optimization strategy. The key idea is to recast and solve the lower level (inner) optimization problem as a multiparametric program, with the parameters corresponding to the optimization variables of the higher level (outer) problem, and then recast the bilevel problem into a set of single-level optimization problems. This strategy can be applied to linear, quadratic, mixed-integer, convex and special classes of nonconvex, nonlinear inner/outer models.

Another important class is the general case of multiparametric mixed-integer linear programming (mp-MILP) problems with parameters involved in both the coefficients of the objective function and the righthand side (RHS) of the linear inequalities, which renders the problem nonlinear and nonconvex. Faísca (2008) recently proposed a very effective algorithm for the solution of such mp-MILP problems, which avoids the need for the global optimization solution of any multiparamet-

Table 2. Milestones in Multiparametric/Explicit Model Predictive Control

Multi–Parametric Model Predictive Control	Pistikopoulos (1997, 2000); Bemporad et al. (2002)
Multi–Parametric Continuous Time Model Predictive Control	Sakizlis (2003)
Hybrid Multi–Parametric Model Predictive Control	Sakizlis (2003)
Robust Multi–Parametric Model Predictive Control	Kakalis (2001); Sakizlis (2003)
Multi–Parametric Dynamic Programming	de la Peña et al. (2004); Faísca (2008)
Multi-Parametric Non-linear Model Predictive control	Johansen (2002); Sakizlis (2003)

ric program. By exploring the model structure, important properties have been established, based on which the global solution of the mp-MILP requires the global solution of a master fixed-point, nonconvex MINLP problem and a slave, multiparametric linear programming (mp-LP) problem.

Future directions

Future research opportunities in multi-parametric programming include

1. *Continuous-time multistage dynamic systems*. Here, important challenges remain for the development of theory and algorithms for the solution of general classes of multiparametric dynamic optimization (mp-DO) problems, also involving 0-1 variables (multiparametric mixed-integer dynamic optimization or mp-MIDO problems). This will require significantly extending and further developing the initial work by Sakizlis (2003). Multistage dynamic systems are also challenging. Here, advances in constrained dynamic programming by multiparametric programming (Faísca, 2008) could provide a good starting point. Extensions to mixed-integer or hybrid dynamic systems constitute another important research direction.
2. *Global optimization/multiparametric programming*. Here, significant progress is expected over the next few years, especially for special classes of nonconvex, nonlinear multiparametric programming problems. Clearly, the important developments that are taking place in the field of global optimization (Tawarmalani and Sahinidis, 2002; Grossmann and Biegler, 2004; Floudas and Gounaris, 2009), are directly applicable and provide excellent starting points for the further development of global optimization solution strategies of general classes of mixed continuous and integer multiparametric programming problems. Furthermore, the development of more effective and accurate solution of general multiparametric nonlinear programming problems is also needed.
3. *Revisiting the fundamentals*. Most/all current algorithms for the solution of multiparametric programming problems employ some form of an active set strategy search, either through the KKT optimality conditions (for time-invariant systems) or the Hamiltonian (for the time-varying dynamic systems), to generate the explicit solutions and corresponding critical regions (Figure 2). Furthermore, any multiparametric algorithm is based on the solution of standard, fixed-point, optimization algorithms, thereby sharing their advantages and disadvantages (for example, degeneracy in linear programming, but also in mp-LP). A grand theoretical challenge remains how to "construct" the explicit solution of a multiparametric programming problem by solving an appropriately defined "minimal" set of auxiliary, fixed-point optimization problems. Here, advances in interior-point methods theory, variational inequalities, and recent results on dynamic optimization and mp-DO, could provide valuable insights.

Multiparametric/Explicit Model Predictive Control Theory—Recent Developments and Future Directions

Past research has mainly focused on the theoretical and algorithmic developments in the area of linear explicit MPC and robust linear explicit MPC, while some initial results in the area of hybrid, continuous-time and nonlinear explicit MPC have also been reported in the open literature (Pistikopoulos et al., 2007b). Some recent developments along with ideas for future research directions are briefly discussed next.

Model order reduction and explicit MPC

The purpose of model order reduction methods is to provide approximate reduced order models (with a reduced number of state variables) for large-scale models. In the case of mp-MPC, reasons for applying model-order reduction methods include insufficient available memory for solving the mp-MPC problem offline, the desire to reduce the time in which the explicit solution of mp-MPC is obtained, and the need to reduce the size of the explicit solution (smaller number of critical regions and smaller number of parameters) in order to speed up both offline and online calculations. In these cases, a reduced order model of the real large-scale process can be directly used for the design of reduced order mp-MPC (Johansen, 2003). However, since reduced order models are only approximations of the real process, the feasibility and optimality of the MPC strategy cannot be generally guaranteed. Narciso (2009) recently developed a systematic method that combines balanced truncation model reduction and mp-MPC design, which obtains the minimum order of the reduced order model, based on which the resulting reduced order mp-MPC controller ensures the feasibility and optimality for the large-scale system.

Explicit nonlinear MPC

Explicit nonlinear MPC (mp-NMPC) refers to the case when the optimization problem of the nonlinear MPC formulation is solved by using mp-NLP methods. Since most real systems are described by nonlinear dynamic models, explicit NMPC has recently started to receive some attention. Recent developments have focused on the following three areas: (1) the linear approximations of the mp-NLP problem, (2) applications of existing mp-NLP methods for solving the optimization problem of the nonlinear MPC, and (3) explicit solutions of the nonlinear continuous-time problem. In Johansen (2002, 2004) an algorithm was presented that uses mp-QP based methods to derive linear piecewise affine approximations to the solution of the explicit NMPC. Sakizlis (2003) introduced a framework for discrete-time explicit NMPC problem, which employs multiparametric global optimization methods (Dua et al., 2004a) to solve the resulting mp-NLP problem. Sakizlis (2003) also presented a comprehensive multiparametric dynamic optimization framework for solving the continuous-time multiparametric nonlinear MPC, which is based on obtaining the exact solution of the continuous-time control variables as nonlinear functions of the time and state variables.

Robust explicit MPC

Robust Explicit MPC refers to methods for the design of explicit controllers for dynamic systems in the presence of bounded disturbances and model uncertainties. Explicit MPC controllers designed with the use of nominal dynamic models cannot in general guarantee feasibility, in terms of constraint satisfaction and system stability, when disturbances and/or model uncertainties are present. The challenge here is to develop algorithms for the design of cost-effective robust explicit MPC controllers, which guarantee constraint feasibility and robust stability for any value of the uncertainty. Recent research here has focused on the design of robust explicit MPC controllers for linear dynamic systems with additive disturbances (in the linear state-space models), and/or parametric model uncertainties (in the system matrices). The case of robust explicit MPC of linear systems with additive disturbances was first examined in the work of Sakizlis (2003). The design of robust explicit MPC for linear systems, with model parametric uncertainties and linear objective functions, was also investigated in Bemporad et al. (2003). Recently, a novel framework for robust explicit MPC of uncertain systems was developed by Pistikopoulos and coworkers (Pistikopoulos et al., 2007b) featuring a robust reformulation/optimization step, a dynamic programming framework for the MPC problem formulation, and a multiparametric programming solution step.

Future directions

Future research opportunities in multiparametric/explicit MPC include

1. *Explicit nonlinear MPC (mp-NMPC)*. While initial and recent developments are quite promising, this is an area that is very much in its early stages. Any developments here will of course depend on fundamental developments in the area of multiparametric nonlinear programming (mp-NLP), multiparametric dynamic optimization (mp-DO), constrained nonlinear dynamic programming by multiparametric programming and global optimization. Furthermore, developments are also expected for the case of nonlinear systems involving 0-1 variables, i.e., hybrid systems—here algorithmic developments for the solution of general mixed-integer nonlinear and dynamic optimization (mp-MILP, mp-MIDO) will be required.

2. *Robust multiparametric/explicit control of hybrid and nonlinear systems*. While recent developments have started to address the design of robust multiparametric controllers for linear MPC systems, there is clearly a need to establish robust controller design methods for general classes of nonlinear systems, within an mp-NMPC framework. Furthermore, theory for the design of robust multiparametric controllers of continuous-time dynamic systems is almost completely lacking. Here, advances in robust optimization of dynamic systems and multiparametric dynamic optimization (mp-DO) under uncertainty are needed. A further challenge arises when 0-1 binary variables are present in the model, i.e., for hybrid systems. Here, for linear systems, recent robust optimization results for MILP models (Lin et al., 2004) offer an excellent starting point; however, there is a lack of theory and algorithms for the robust solution of general mixed-integer nonlinear and dynamic optimization problems involving uncertainty.

3. *Model reduction/approximation—system identification—estimation*. The further development and application of model reduction techniques and/or approximations constitute another important research direction. Here, a key objective is to derive model instances from which multiparametric/explicit MPC controllers can be readily designed based on state-of-the-art multiparametric programming algorithms. While initial efforts have focused on linear systems, a much more formidable task is the efficient utilization and further development of nonlinear model reduction strategies. A further challenge arises while dealing with uncertainty and disturbances—here robust system identification techniques can be of relevance. Furthermore, estimation techniques will play a key role toward the development of robust explicit MPC strategies—initial developments (Sakizlis, 2003; Darby and Nikolaou, 2007) provide a good starting point.

MPC-on-a-chip applications—Recent Developments and Future Directions

Three previous applications, that clearly demonstrated the potential of the MPC-on-a-chip technology are, (1) a nitrogen generator/air separator plant (Mandler et al., 2006), (2) an active valve train control system (Kosmidis et al., 2006), and (3) a pilot plant PARSEX reactor (Dua P., 2005). Biomedical systems and devices are another application area, where MPC-on-a-chip technology can bring distinct benefits, as shown in recent work for insulin delivery for type 1 diabetes, anesthesia, and chemotherapeutical agents (Dua P., 2005). The ability to derive offline and in an explicit form the control performance of the device, under simulated/patient characteristics allows for enhanced safety and increased understanding of the preclinical/clinical, approval and actual implementation stages. This is the subject of the MOBILE project, a recently awarded ERC (European Research Council) Advanced Grant (Modelling, Control and Optimization of Biomedical Systems, ERC, Advanced Grant, 2009-2013), see also Figure 3.

Power/electricity generation systems (fuel cells) with storage devices (for example hydrogen storage) especially for small-scale applications (in-house, portable) are another important application area, where the MPC-on-a-chip technology can play an important role (Georgiadis et al., 2009). Examples of other current applications include a hybrid membrane/pressure swing adsorption (PSA) separation system (EU Project HY2SEPS), unmanned air vehicles (UAV) (EPSRC project EP/E047017/1); the Imperial Racing Green Car Project (IRG) 2009, and batch control (EU Project CONNECT).

A much wider classification of the type of applications and the opportunities for the MPC-on-a-chip technology is as follows. Broadly speaking, application areas can be divided into three types:

- Type 1, corresponding to large scale, expensive industrial processes with slow/medium dynamics.
- Type 2, related to medium scale and cost processes/equipment with medium/fast dynamics.
- Type 3, corresponding to small scale, relatively inexpensive processes/equipment with medium/fast dynamics.

Examples of Type 1 applications include large-scale chemical and manufacturing process industries (oil refineries, chemical plants, etc.). Such systems most likely have already invested in sufficient control hardware and software infrastructure—here the benefits of advanced control will primarily arise from the implementation of traditional MPC/online optimization technology (Qin and Badgwell, 2003). Explicit MPC, mainly in software form, can play some role, especially

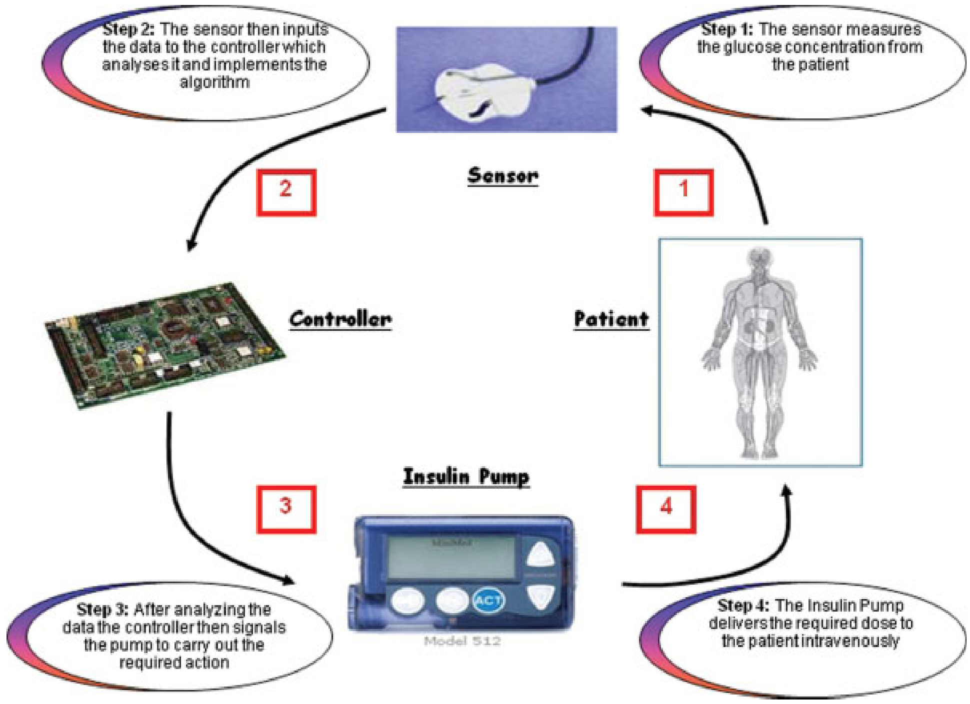


Figure 3. MPC-on-a-chip technology in biomedical systems and devices.

at the low-level process control, essentially substituting PI(D) systems and/or empowering online optimization, by an intelligent implementation of multiparametric programming concepts in the online dynamic optimization code (Biegler and coworkers (Zaval et al., 2008), Bock and coworkers (Ferreau et al., 2008), and Marquardt and coworkers (Kadam et al., 2007)).

Examples of Type 2 applications include systems such as small air separation plants, PSA units, unmanned air vehicles control and the like. Such systems usually have some available control hardware and software infrastructure, but either this is rather limited and often not suitable for full-scale online MPC or other considerations (such as speed, power requirements and cost) may pose significant constraints. For such systems, the MPC-on-a-chip technology can be an ideal choice, as the advanced control technology that is most affordable (in size, cost, computational speed), as proven in actual previous implementations (Mandler et al., 2006; Kosmidis et al., 2006; Dua P., 2005).

Examples of Type 3 applications, the most common in real life, include embedded type of systems (all-in-one), such as portable devices, power generators, household devices and the like. In such systems, the available control hardware infrastructure is usually based on microchip technology, with the available and limited computational power not able to support any online optimization activity. For such systems, the MPC-on-a-chip technology is essential, most suitable and perhaps the only credible alternative for an advanced control implementation. It is also highly flexible and can be easily reprogrammed in new control hardware (FPGA, wireless), as being demonstrated in the Imperial Racing Green Car project.

A Framework for Multiparametric Programming and Explicit MPC

Based on the past and recent advances in multi-parametric programming and multi parametric/explicit MPC, and the ability to arrive at MPC-on-a-chip advanced control solutions, as briefly described in the previous sections, here we present a comprehensive framework for the systematic development and design of validated multiparametric controllers, featuring four major steps.

Step 1 involves the development of a suitable, usually high-fidelity dynamic model, to provide a detailed description of the process/system/equipment. Here, gPROMS® (PSE, Ltd., 2008) and its suite of tools, will be an ideal starting point—provision should be made for increased accuracy and validation of such mathematical models. Step 2 then involves model approximation. This step is important in order to arrive at a suitable MPC formulation, which can then be readily solved by multiparametric programming techniques. Here, advances in (nonlinear) model-order reduction will play a key role, as well as advanced and robust system identification methods, as discussed earlier. Step 3 corresponds to the design of the multiparametric/explicit model predictive controller, by applying the available methods of multiparametric programming and control presented earlier—based on an approximate model of the system. A family of explicit (and robust) controllers can be designed at this stage, if alternative model approximations are available in Step 2. Step 4 finally involves the offline validation of the derived multiparametric/explicit controller(s). This is done by incorporating the controller (i.e., its corresponding control law expressions) into the high-fidelity model and perform computational studies (simulation and dynamic optimization), to test the controller's performance. Note that as the controller is based on an approximate model, possible deviations from the desired target behavior may be detected at this step and the controller may not be accepted—with the procedure to be repeated until desired behavior is achieved for the chosen explicit controller. Multiple controller designs can be also tested in parallel at this step.

All these tasks/steps are performed computationally offline and before any real implementation onto the real system. The potential of the proposed framework is that it can lead to the design of "tailored-made" robust explicit model predictive controllers, which can be fully validated and tested offline, thereby significantly reducing the cost and time of testing as well as the risk of failing at the actual online installation. This also opens possibilities to link the proposed framework with simultaneous process and control design optimization strategies.

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